COMPLETE RESPONSE OF AN SDOF SYSTEM WITH A MIXED DAMPING MODEL

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SUMMARY: The hysteretic damping model has been used for more than half a century. For many structures, mainly metallic ones, it gives a simple representation for forced, steady-state vibrations, usually showing a remarkable agreement with the experimental results.
However, when dealing with free or transient vibrations, mathematical difficulties have been raised by several authors, concerning apparent non-causal behaviour of the model.
In a series of recent papers, the authors have shown that such a model can be used without incurring in such non-causal behaviour and that this same model could show a very good agreement with experimental results also for free vibration. Moreover, the approach was shown to be applicable to the viscous damping model too. As a consequence, it becomes possible to study a mixed model with both hysteretic and viscous terms.
The present paper encompasses the results of those previous papers and includes an example to illustrate the correlation between theoretical and experimental results.

KEYWORDS: mixed damping, complete response, hysteretic damping, viscous damping

1. INTRODUCTION

The problem of modelling the dissipation of energy in vibrating solids, namely in metals, has been around for quite a while. At least since the middle of the XX century several authors have approached the issue. In 1949, Soroka [1] and in 1952, Miklestad [2] tried to use complex stiffness to model the behaviour of structural elastic and dissipative characteristics of materials. Although those authors and many others have addressed the problem, some inconsistencies delayed the formulation of a satisfactory solution. The main problems arose when trying to obtain the homogeneous solution for the governing differential equation. Non-causality problems have been reported by other authors [3] who used the real initial conditions as the initial values for the differentiate variable. In 2005, the authors presented a paper [4] where a new approach to the problem provided a solution that showed very good agreement to experimental data. Such an approach was further developed and explored in subsequent papers to address the transient response to a harmonic excitation [5] and to include mixed damping – hysteretic and viscous [6]. That approach recognized that the observable displacements at any time – including the real initial conditions at time zero – were the real part of a complex variable. As a consequence, no causality problems arise, as long as that real part is concerned.
The present work uses the same approach in order to obtain the complete solution for a system with both viscous and hysteretic damping.
It must be stressed out that this approach endeavours to find a coherent mathematical model consistent with the physical reality. The damping effect observed in structures, especially metallic ones, is practically independent of the frequency (for the range between 10Hz and 2000Hz) [7]. This effect has been related to the hysteresis exhibited by materials and is therefore called hysteretic damping. When viscous damping is also present, both effects should be taken into account when modelling the system. Also, in non-metallic materials, such as polymers, it is reasonable to assume that both viscous and hysteretic damping mechanisms are present. This reasoning is the driver for the present work: to find a mathematical formulation that allows for the inclusion of both types of damping in the equation in a closed form solution consistent with the physically observable reality.

2. THEORETICAL DEVELOPMENT

The theoretical approach assumes the governing differential equation for an s.d.o.f. system with hysteretic and viscous damping, as:

$$m\ddot{x} + c\dot{x} + k^{*}x = f(t) \tag{1}$$

where m is the mass, c is the viscous damping coefficient and k^{*} is the complex stiffness. In such a context, x should be understood, not as a displacement, but as a variable whose real part is the measurable displacement u, that is:

$$u = \Re(x) \tag{2}$$

The exciting force $f(t)$ here considered will be a harmonic force of (possibly complex) amplitude F such as:

$$f(t) = Fe^{i\omega t} \tag{3}$$

Considering:

$$\begin{cases} k^{*} = k(1 + i\eta) \\ \omega_n = \sqrt{\dfrac{k}{m}} \\ \xi = \dfrac{c}{2m\omega_n} \end{cases} \tag{4}$$

Equation (1) can be written as:

$$\ddot{x} + 2\omega_n\xi\dot{x} + \omega_n^2(1 + i\eta)x = \frac{Fe^{i\omega t}}{m} \tag{5}$$

The complete solution for equation (5) is the sum of its homogeneous and particular solutions. The homogeneous solution can be found using the Laplace transform, with $x_h(t)=Ce^{st}$, leading to:

$$s^2 + 2\omega_n\xi s + \omega_n^2(1 + i\eta) = 0 \tag{6}$$

From which the solutions may be found:

$$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1 - i\eta} \tag{7}$$

In order to uncouple the real and imaginary parts of $\boldsymbol{s}$, one can rewrite (7) as:

$$s = -\xi\omega_n \pm \omega_n(a + ib) \tag{8}$$

where:

$$\begin{cases} a = +\sqrt{\dfrac{-1 + \xi^2 + \sqrt{\left(1 - \xi^2\right)^2 + \eta^2}}{2}} \\ b = +\sqrt{\dfrac{+1 - \xi^2 + \sqrt{\left(1 - \xi^2\right)^2 + \eta^2}}{2}} \end{cases} \tag{9}$$

The homogeneous solution $x_h(t)$ can therefore be written as:

$$x_h(t) = C_1 e^{(-\xi + a)\omega_n t} e^{+ib\omega_n} + C_2 e^{(-\xi - a)\omega_n t} e^{-ib\omega_n t} \tag{10}$$

where C_1 and C_2 are complex constants.
Now, the constants C_1 and C_2 must be evaluated in such a way that the physical conditions for the problem are satisfied. These conditions include, besides the usual initial conditions of displacement and velocity, the assumption that the system is not getting energy but dissipating it through both viscosity and hysteresis; since any non-zero value of C_1 would allow for the infringement of this last requirement, its value must be zero and, considering $C_2 = C$, for the sake of simplicity, the solution becomes:

$$x_h(t) = Ce^{-(\xi+a)\omega_n t} e^{+ib\omega_n t} \tag{11}$$

Defining:

$$\begin{cases} \omega_d = b\omega_n \\ \alpha = (\xi + a)\omega_n \end{cases} \tag{12}$$

where ω_d is the damped natural frequency and α the decaying rate; equation (11) becomes:

$$x_h(t) = Ce^{-\alpha t} e^{i\omega_d t} \tag{13}$$

As for the particular solution of equation (5), it may be written as usual:

$$x_p(t) = Xe^{i\omega t} \tag{14}$$

where X is a complex amplitude given by:

$$X = \frac{F/m}{\omega_n^2 - \omega^2 + i\left(\omega_n^2\eta + 2\xi\omega_n\omega\right)} \tag{15}$$

The complete solution is, therefore:

$$x(t) = Ce^{-\alpha t} e^{i\omega_d t} + Xe^{i\omega t} \tag{16}$$

Considering now equation (2), the physical initial conditions for displacement (x_0) and velocity (v_0) may be written as:

$$\begin{cases} u(0) = x_0 \\ \dot{u}(0) = v_0 \end{cases} \tag{17}$$

Also, writing the complex amplitude of the particular response from equation (14) as the sum of its real and imaginary parts

$$X = X_r + iX_i \tag{18}$$

Differentiating equation (16) and using it with the result and with equations (2), (17) and (18), one can write:

$$\begin{cases} x_0 = C_r + X_r \\ v_0 = -(\alpha C_r + \omega_d C_i) - \omega X_i \end{cases} \tag{19}$$

After some simple manipulations, it is possible to obtain the value for the complex constant:

$$C = x_0 - X_r - i\frac{v_0 + \omega X_i + \alpha(x_0 - X_r)}{\omega_d} \tag{20}$$

and the analytical form for the complete solution becomes:

$$x(t) = \left(x_0 - X_r - i\frac{v_0 + \omega X_i + \alpha(x_0 - X_r)}{\omega_d} \right) e^{-\alpha t} e^{i\omega_d t} + Xe^{i\omega t} \tag{21}$$

According to equation (2), the real part of x is the measurable displacement $u(t)$:

$$u(t) = e^{-\alpha t}\left((x_0 - X_r)\cos(\omega_d t) + \frac{v_0 + \omega X_i + \alpha(x_0 - X_r)}{\omega_d}\sin(\omega_d t) \right) + X\cos(\omega t) \tag{22}$$

3. DISCUSSION

It should be stressed that the above formulations are valid for the entire range of non-negative values for both the viscous and hysteretic damping factors. As such, the classical distinction among critical, sub-critical and super-critical damping becomes a matter of simple convenience of speaking and do not require different analytical expressions, let alone essentially different formulations.

For any general case, within the scope of linear s.d.o.f. systems with viscous and hysteretic damping the complete measurable time response is given by equation (22). The following discussion regards the effect of the involved parameters on the physical behaviour of the system. Equation (22) shows that such a behaviour can be assessed from the knowledge of the measurable initial conditions, the (complex) applied load and the parameters of the system. The result is, as expected, the superposition of an exponentially decaying sine-wave with frequency ω_d and another sine with the same frequency ω of the applied load. The aspects that this model brings forward are the object of this discussion, namely the effect of the damping on the effective free response, ω_d and the attenuation ratio, α, which can be observed from equation (16), taking into account the definitions in equations (9) and (12).

3.1. Damped frequency

From equation (12) factor b establishes the relation between the effective frequency of the free response and the undamped natural frequency of the system. Recalling equation (9)

$$b = +\sqrt{\frac{+1 - \xi^2 + \sqrt{(1-\xi^2)^2 + \eta^2}}{2}} \tag{23}$$

an easier way to visualize the effect of both viscous and hysteretic factors on the frequency of the response is to plot a graphic of b against ξ and η. Fig. 1 shows such a plot.

Although the hysteretic factor seldom attains orders of magnitude larger that 1%, the plot was extended into much larger values, so the effect of hysteretic damping can be made visible. In fact, for common values of η, the absolute change it brings to the frequency is almost negligible. However, the relative effect is not: the slight rise in the frequency due to this factor, which is enough to explain a residual oscillating behaviour that real systems exhibit even for very high viscous damping factors. It is clear from Fig. 1 that the frequency becomes zero for values of ξ greater than unity, only for η=0. As it is, the so called critical damping factor only makes sense for systems without hysteretic damping.

It must be nevertheless stressed that, from the practical point of view, not only this effect is very small, for usual values of η, but also the values of b are very small for values of ξ greater than unity.

A further remark concerns the evolution of b against ξ for null η, which follows the same pattern predicted by the classical approach but encompassing both sub- and super-critical damping within a single analytical expression.

Due attention is also required to the fact that an increase in the hysteretic damping factor increases the free response frequency. Some authors consider this increase counter intuitive [2], but a simple reasoning may help to dissipate the misunderstanding: an increasing damping effect implies an increasing loss of energy per cycle; while the viscous model translates such loss to decreased frequency and amplitude for the following cycle, the hysteretic model presents only a marked decrease in amplitude (see next section), scarcely mitigated by a nearly negligible increase in frequency. The net effect of any damping shall be understood as a loss of energy per cycle and only incidentally as per time, since that is the common ground for both models.

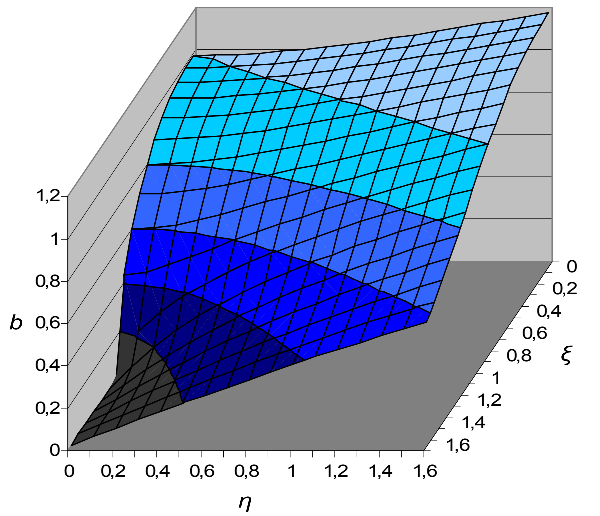

Fig. 1 – Relation b between the effective frequency of the free response and the natural frequency of the system.

3.2. Attenuation ratio

The quantity α defined in equation (12) stands for the attenuation of the free response part: the larger it is, the fastest the transient response vanishes. From equations (9) and (12), this parameter may be given by:

$$\alpha = \left(\xi + \sqrt{\frac{-1+\xi^2+\sqrt{\left(1-\xi^2\right)^2+\eta^2}}{2}} \right)\omega_n \tag{24}$$

The results, plotted in Fig. 2, show what was expected: an increase of α with both damping factors. The only small remarks worth making are:

- the sharper change of α with ξ for values just above unity, when η is very small;
- the fact that the above effect is unparalleled along the η axis, since the evolution is quasi-linear;
- and the smoothing of that sharp change with ξ for increasing values of η.

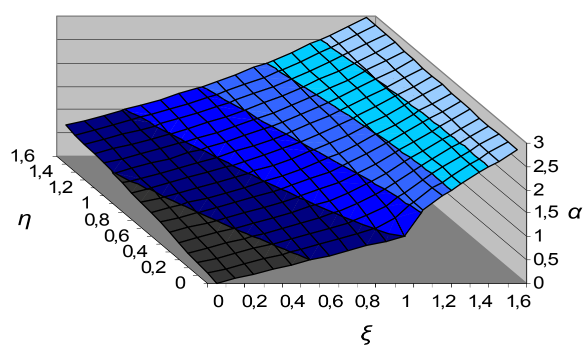

Fig. 2 – Attenuation factor α against ξ and η.

This factor can be related to the free vibration logarithmic decrement presented elsewhere [6]:

$$\delta = 2\pi \frac{(\xi + a)}{b} \tag{25}$$

And this, plotted graphically gives:

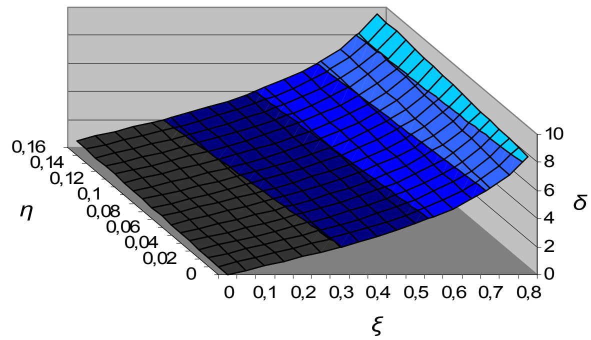

Fig. 3 – Influence of the hysteretic and damping factors on the logarithmic decrement

4. EXPERIMENTAL EXAMPLE

To illustrate and validate the use of the mixed model presented above, a simple experimental setup was used, as drafted in Fig. 4. A brass beam was cantilevered and its free end was loaded with a rigid mass and connected to a dashpot damper, so that both the hysteretic and viscous damping factors came into play. The fluid in the dashpot was a low viscosity so that the viscous factor would not be so much larger than the hysteretic factor as to mask it. The plan was to apply a sine-wave force to the free end and measure the complete response.

This measurement of the complete response, however, was not conclusive because the start-up of the shaker used presented such irregularities that prevented the use of the initial time segment and, after that, only the steady-state vibration was measurable.

To circumvent this problem and since the steady-state model presents nothing really new, the experiment focused on the free response with mixed damping. The aim was to show that the model presented above can accurately predict the free response of the system.

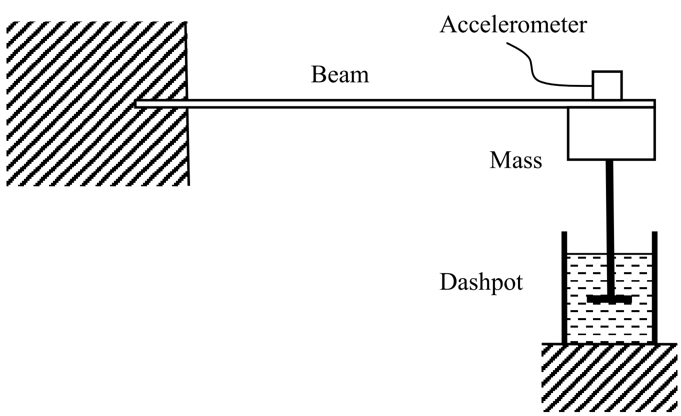

Fig. 4 – Experimental setup for mixed damping tests (scheme)

The tests were carried out giving an initial displacement to the free end of the beam, letting the system vibrate freely and recording the time response.

A first test was performed without any liquid in the dashpot. From the results, presented in Fig. 5, the hysteretic damping factor was obtained using equations (25) and (9), with ξ=0.

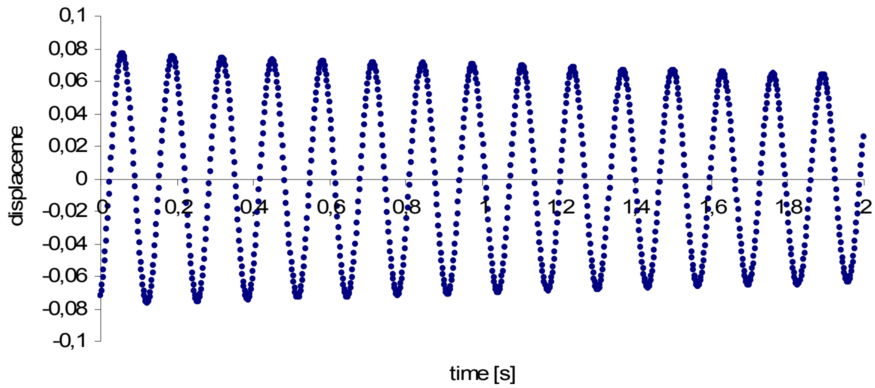

Fig. 5 – Time story with hysteretic damping only

The value obtained for the hysteretic damping factor was η=0.005.
A second test was then performed with water in the dashpot, so the resulting damping was both hysteretic and viscous (Fig. 6).

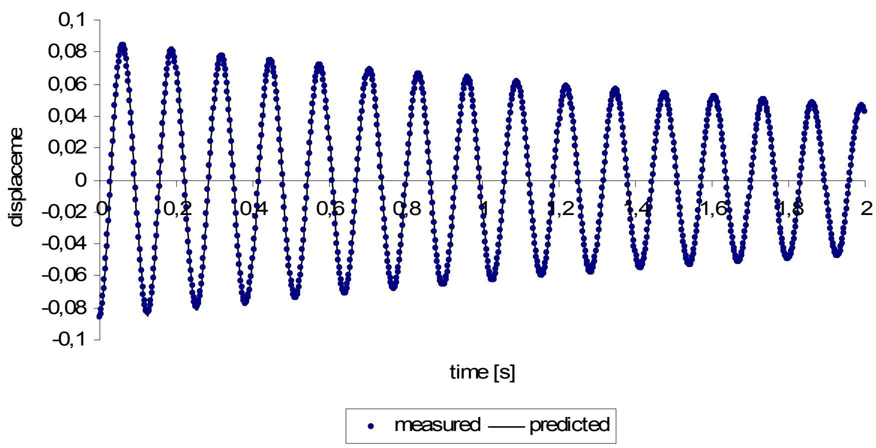

Fig. 6 – Time story for mixed damping:

Equation (25) allowed to find the viscous damping factor ξ=0.004. Equation (19), with X=0, was used to obtain the complex initial conditions and the complex amplitude C=-0.086-0.020i to be used into equation (22). The predicted results were plotted also in Fig. 6. The match was almost perfect.

5. CONCLUSIONS

A coherent formulation for the complete response of s.d.o.f. systems with both hysteretic and viscous damping was presented. It sums up previous papers that address the issue of the free response for the hysteretic damping model. The complete solution for a sine-wave load was presented as a result of extending the concept of complex response from the forced to the transient response. Using this formulation, the measurable response is taken as the real part of a complex function.
The behaviour of this model was briefly discussed and the effect it predicts on the damped natural frequency, the attenuation factor and the logarithmic decrement.
An experimental example illustrated the use of the formulation for the free response with mixed damping.

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